Nuclear Size and Magnetic Effects in the Radiative Tail of Electron-Nucleus Scattering*

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The spectrum of scattered electrons due to bremsstrahlung from static nuclear charge and magnetic-moment distributions is calculated in Born approximation. As expected, large reductions from point-nucleus cross sections are obtained for scattering involving large momentum transfers, and magnetic bremsstrahlung is important in the same large-angle region where magnetic elastic scattering is appreciable. Integrations were carried out numerically on an IBM-7090 computer for some representative cases using form factors from elastic electron scattering. The results may be used in interpreting inelastic electron-scattering experiments where one wishes to distinguish the radiative tail of the elastic peak from inelastic nuclear processes.

I. INTRODUCTION

EXCITATION of nuclei by the inelastic scattering of high-energy electrons is a useful technique for the investigation of nuclear structure.[1] The process is ordinarily detected by observing the spectrum of the scattered electrons. This requires knowledge of the spectra from competing mechanisms for inelastic scattering of electrons, since they must be unfolded from the observed spectrum. The purpose of this paper is to discuss one of the fundamental background processes, namely, the radiative tail of elastic electron scattering due to bremsstrahlung.

Our concern here is to explore how nuclear effects modify the well-known Bethe-Heitler results for bremsstrahlung in a point-charge Coulomb field. We describe the nucleus by arbitrary static charge and magnetic-moment distributions (or by their Fourier transforms, the form factors F_{ch} and F_{mag}). This procedure neglects bremsstrahlung processes in which the nucleus is left in an excited state.[2]

In Sec. II we calculate the totally differential cross section for bremsstrahlung using the Born approximation, i.e., keeping the lowest order nonvanishing terms in the external electromagnetic field of the nucleus. The spectrum of the scattered electrons may then be obtained from a single integration which, owing to the presence of the form factors F_{ch} and F_{mag}, must, in general, be done numerically. Analytic results for the spectrum from a point charge and point magnetic moment are discussed. In Sec. III we give some representative numerical results displaying the effects of finite nuclear size and magnetic-moment distribution on the electron spectrum.

It is known from elastic scattering of high-energy electrons that the use of the Born approximation is justified for light nuclei, except when the momentum transfer is near a zero of the form factors.[3] Similar remarks apply to the radiative tail from elastic scattering; large distortions are introduced into the totally differential cross section in the region of these diffraction minima. Usually two particular momentum transfers give a large contribution to the integral over the totally differential cross section; provided neither of these coincides with a zero of the form factors, the Born-approximation result for the electron spectrum may be trusted. In somewhat heavier elements, by analogy with the elastic case, one will not trust the Born approximation at a given angle beyond the incident energy for which the dominant momentum transfers first approach a zero of the form factors ($q\lesssim 1\ \mathrm{F}^{-1}$). This is probably the most serious limitation in the present work.

Some further features which we neglect in the present calculation should be mentioned: (i) higher order terms in the electromagnetic field (as distinct from the external field), (ii) screening of the nucleus by atomic electrons, and (iii) nuclear recoil. The first of these is at most a fraction of a percent.[4] Screening is not important for high-energy electron scattering except at exceedingly small angles[5] [$\sin^2\frac{1}{2}\Theta$ of the order of $(Z\alpha k/4E_0E)^{1/2}$ where the notation is defined below] due to the comparatively large momentum transfers involved.[6] The corrections to the Bethe-Heitler equation due to recoil have been discussed by Drell[7] and we expect that similar

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[1] W. C. Barber, Ann. Rev. Nucl. Sci. **12**, 1 (1962); D. B. Isabelle and G. R. Bishop, Nucl. Phys. **45**, 209 (1963).

[2] The radiative tail from an excited nuclear level is usually assumed to have the same shape as the elastic case, and for excitation energies less than 30 MeV, where a few separate inelastic peaks contribute, the radiative tail from the elastic peak dominates. An iterative procedure for calculating the radiative tail from such inelastic process, given sufficient experimental data, has been proposed by several authors, e.g., Y. S. Tsai, *Nucleon Structure: Proceedings of the 1963 Conference at Stanford University*, edited by R. Hofstadter and L. I. Schiff (Stanford University Press, to be published); D. B. Isabelle and G. R. Bishop, Ref. 1.

[3] R. Herman and R. Hofstadter, *High Energy Electron Scattering Tables* (Stanford University Press, Stanford, California, 1960), pp. 8–13.

[4] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., p. 254.

[5] The argument follows Heitler, Ref. 4, Sec. 25.3.

[6] On the other hand, screening may be very important when the bremsstrahlung spectrum is considered; one then introduces atomic form factors in the usual manner.

[7] S. D. Drell, Phys. Rev. **87**, 753 (1952).

corrections apply to the magnetic bremsstrahlung. These corrections are believed to be a less serious source of error than the use of the Born approximation, except perhaps for the very lightest nuclei.

After the completion of this work a related calculation by Maximon and Isabelle[8] was brought to our attention. These authors are also concerned with the radiative tail from elastic electron scattering. However, they do not discuss magnetic effects, which are necessary for the large-angle electron-scattering experiments which motivated the present work, and they focus their main attention on the rederivation of the high-energy limit expressions of Schiff and the corrections thereto.

II. ANALYTICAL RESULTS

The differential cross section for bremsstrahlung from static charge and magnetic-moment distributions, after summing over final electron and nuclear spins and photon polarizations, and averaging over initial electron and nuclear spins, is found to be

$$d\sigma=\frac{\bar{\phi}}{(2\pi)^2}\frac{p}{p_0}\frac{dk}{k}\frac{d\Omega_p d\Omega_k}{q^4}\left\{|F_{\mathrm{ch}}|^2T_{\mathrm{ch}}+\left(\frac{\mu}{Ze}\right)^2\left(\frac{I+1}{3I}\right)q^2|F_{\mathrm{mag}}|^2T_{\mathrm{mag}}\right\}, \tag{1}$$

where

$$T_{\mathrm{ch}}=\frac{(4E^2-q^2)(\mathbf{k}\times\mathbf{p}_0)^2}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)^2}+\frac{(4E_0^2-q^2)(\mathbf{k}\times\mathbf{p})^2}{(kE-\mathbf{k}\cdot\mathbf{p})^2}+2\frac{k^2(\mathbf{k}\times\mathbf{p})^2+k^2(\mathbf{k}\times\mathbf{p}_0)^2-(4EE_0+2k^2-q^2)(\mathbf{k}\times\mathbf{p})\cdot(\mathbf{k}\times\mathbf{p}_0)}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)(kE-\mathbf{k}\cdot\mathbf{p})}, \tag{2}$$

and

$$T_{\mathrm{mag}}=\frac{(4p^2+q^2)(\mathbf{k}\times\mathbf{p}_0)^2}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)^2}+\frac{(4p_0^2+q^2)(\mathbf{k}\times\mathbf{p})^2}{(kE-\mathbf{k}\cdot\mathbf{p})^2}+4k^2\left(\frac{kE_0-\mathbf{k}\cdot\mathbf{p}_0}{kE-\mathbf{k}\cdot\mathbf{p}}+\frac{kE-\mathbf{k}\cdot\mathbf{p}}{kE_0-\mathbf{k}\cdot\mathbf{p}_0}\right)$$
$$+2\frac{k^2(\mathbf{k}\times\mathbf{p})^2+k^2(\mathbf{k}\times\mathbf{p}_0)^2-[4(EE_0-1)+2k^2+q^2](\mathbf{k}\times\mathbf{p})\cdot(\mathbf{k}\times\mathbf{p}_0)}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)(kE-\mathbf{k}\cdot\mathbf{p})}. \tag{3}$$

We use the notation: units $\hbar=c=m_e=1$; initial electron momentum and energy: $\mathbf{p}_0$, E_0; scattered electron momentum and energy: $\mathbf{p}$, E; photon momentum and energy: $\mathbf{k}$, $k\equiv E_0-E$; momentum transfer: $\mathbf{q}=\mathbf{p}_0-\mathbf{p}-\mathbf{k}$; $\bar{\phi}=Z^2r_0^2/137\simeq Z^2 5.794\times10^{-28}$ cm^2 (r_0=classical electron radius); nuclear spin and magnetic moment: I, $\mu=\lambda(e/2M_p)$.

T_{ch} is just the familiar expression from the Bethe–Heitler equation; T_{mag} is a very similar looking but essentially different expression.

The form factors F_{ch} and F_{mag}, arising from the nuclear charge and magnetic-moment distributions $\rho(\mathbf{r})$ and $\boldsymbol{\mu}(\mathbf{r})=\mu\mathbf{I}/I$, respectively, enter into the calculation as follows. The external potential, in momentum space, is given by[9]

$$A_0(\mathbf{q})=(2\pi)^{-3}q^{-2}\int\rho(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \tag{4}$$

$$\mathbf{A}(\mathbf{q})=(2\pi)^{-3}q^{-2}\int\mathbf{j}(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \tag{5}$$

where the magnetization current density is written as $\nabla\times\boldsymbol{\mu}(\mathbf{r})=\mathbf{j}(\mathbf{r})$. Using the identity $\nabla\times(\boldsymbol{\mu}e^{i\mathbf{q}\cdot\mathbf{r}})=i\mathbf{q}\times\boldsymbol{\mu}e^{i\mathbf{q}\cdot\mathbf{r}}+\mathbf{j}e^{i\mathbf{q}\cdot\mathbf{r}}$, Eq. (5) may be transformed into

$$\mathbf{A}(\mathbf{q})=i(2\pi)^{-3}\frac{\mathbf{I}\times\mathbf{q}}{Iq^2}\int\mu(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \tag{6}$$

where we have written the second integral as a surface integral which vanishes. Equations (4) and (6) are just the potentials due to a point charge and magnetic moment multiplied by the form factors for the nuclear charge and magnetic-moment distributions, respectively,

$$F_{\mathrm{ch}}(\mathbf{q})=(Ze)^{-1}\int\rho(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \tag{7}$$

$$F_{\mathrm{mag}}(\mathbf{q})=i\mu^{-1}\int\mu(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}. \tag{8}$$

There are no cross terms in Eq. (1) between the scalar and vector potentials because of the nuclear spin sums. If the nuclear charge and magnetic-moment distributions are spherically symmetric the form factors will be functions of q^2 only; in the subsequent discussion we will assume that this is the case.

It can be shown that the form of Eq. (1) is quite general, i.e., that any elastic or inelastic process occurring in electron scattering which involves one-photon exchange only and in which the nuclear states are not observed depends on two form factors (which in general

[8] L. C. Maximon and D. B. Isabelle, Phys. Rev. 133, B1344 (1964). We are grateful to Professor Barber for informing us of this work.

[9] P. V. C. Hough, Phys. Rev. 74, 80 (1948).

are functions of momentum and energy transfer).[10] The same form factors given by Eq. (7) and (8) enter into the cross section for elastic electron scattering. A simple calculation shows that for elastic scattering from the external field given by Eq. (4) and (5) the cross section is

$$\frac{d\sigma}{d\Omega}=\frac{Z^2r_0^2}{4}\frac{(1-v^2)}{v^4\sin^4\frac{1}{2}\Theta}\left\{|F_{\rm ch}|^2(1-v^2\sin^2\tfrac{1}{2}\Theta)+\left(\frac{\mu}{Ze}\right)^2\left(\frac{I+1}{3I}\right)|F_{\rm mag}|^2q^2v^2(1+\sin^2\tfrac{1}{2}\Theta)\right\}, \quad (9)$$

where $v=p_0/E_0$ and $q^2=4p_0^2\sin^2\frac{1}{2}\Theta$. For $v\simeq1$, this becomes[11]

$$\frac{d\sigma}{d\Omega}=\frac{Z^2r_0^2}{4E_0^2}\frac{\cos^2\frac{1}{2}\Theta}{\sin^4\frac{1}{2}\Theta}\left\{|F_{\rm ch}|^2+\left(\frac{\mu}{Ze}\right)^2\left(\frac{I+1}{3I}\right)\times|F_{\rm mag}|^2q^2(1+2\tan^2\tfrac{1}{2}\Theta)\right\}, \quad (10)$$

which except for the proper kinematic factors (i.e., neglecting recoil) is the Rosenbluth-type formula for elastic electron-nucleus scattering. Thus we can identify[12] $|F_{\rm ch}|^2$ and $|F_{\rm mag}|^2$ (in the limit $q/M\ll1$) with the experimentally determined form factors from elastic electron scattering.

In order to calculate the distribution in energy and angles of the scattered electrons it is necessary to integrate Eq. (1) over photon angles $d\Omega_k$. This integration was first performed by Racah[13] starting from the Bethe-Heitler equation, and the calculation was later repeated by McCormick *et al.*,[14] who corrected some misprints and also gave a useful approximation for high energies and large angles. An approximate integration of the Bethe-Heitler equation was performed by Schiff,[15] who considered only the bremsstrahlung emitted parallel to either the incident or scattered electron; Schiff's result is easily generalized to include the effect of a finite-sized charge distribution.[16] This approximation does not apply to situations where the elastic scattering is small, that is for scattering angles greater than 160°.[17] Since Eq. (1) contains essentially arbitrary functions of q^2, we employ a transformation similar to that of Rawitscher[18] to transform an angular integration into one over q^2. In a coordinate system with z axis parallel to the fixed vector $\mathbf{a}\equiv\mathbf{p}_0-\mathbf{p}$ and with azimuth φ measured from the plane of $\mathbf{a}$ and $\mathbf{p}_0$, $d\Omega_k\to(ak)^{-1}dxd\varphi$, where $x\equiv\frac{1}{2}q^2$. The

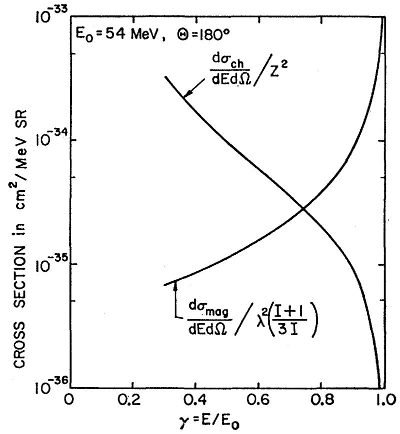

FIG. 1. Relative contributions to the radiative tail of the elastic peak from a point charge (Ze) and point magnetic moment $(\lambda e/2M_p)$ in the limit E_0, $E\gg1$, $\Theta\gg E_0^{-1}$.

φ integration can be performed analytically and the result expressed as

$$\frac{d\sigma}{dEd\Omega}=\frac{\bar{\Phi}}{4\pi}\frac{p}{p_0}\int_{x_{\rm min}}^{x_{\rm max}}\frac{dx}{x^2}\left\{|F_{\rm ch}|^2R_{\rm ch}+\left(\frac{\mu}{Ze}\right)^2\left(\frac{I+1}{3I}\right)2x|F_{\rm mag}|^2R_{\rm mag}\right\}, \quad (11)$$

where

$$\begin{aligned}R_{\rm ch}=&-2a^{-1}-(2E_0^2-x)(\alpha\beta_0+x\beta)X^{-3/2}\\&-(2E^2-x)(\alpha\beta+x\beta_0)X_0^{-3/2}+\{2(E_0^2+E^2)+\alpha\\&-x-2[(1+\alpha)(E_0^2+E^2-x)-k^2](\alpha-x)^{-1}\}\\&\times(X^{-1/2}-X_0^{-1/2}), \quad (12)\end{aligned}$$

$$\begin{aligned}R_{\rm mag}=&\,2a^{-1}-(2p_0^2+x)(\alpha\beta_0+x\beta)X^{-3/2}\\&-(2p^2+x)(\alpha\beta+x\beta_0)X_0^{-3/2}+\{2(E_0^2+E^2)-\alpha\\&+x-2[(1+\alpha)(p_0^2+p^2+x)-k^2](\alpha-x)^{-1}\}\\&\times(X^{-1/2}-X_0^{-1/2}), \quad (13)\end{aligned}$$

and where

$$\begin{aligned}x_{\rm min}&=\tfrac{1}{2}q_{\rm min}^2=\tfrac{1}{2}(a-k)^2,\\x_{\rm max}&=\tfrac{1}{2}q_{\rm max}^2=\tfrac{1}{2}(a+k)^2,\\\alpha&=EE_0-\mathbf{p}\cdot\mathbf{p}_0-1,\\\beta_0&=\mathbf{a}\cdot\mathbf{p}_0=p_0^2-\mathbf{p}_0\cdot\mathbf{p},\\\beta&=\mathbf{a}\cdot\mathbf{p}=\mathbf{p}_0\cdot\mathbf{p}-p^2,\\X_0&=p_0^2x^2+2x[k^2-\alpha(EE_0-1)]+p^2\alpha^2,\\X&=p^2x^2+2x[k^2-\alpha(EE_0-1)]+p_0^2\alpha^2.\end{aligned}$$

Equation (11) is the exact Born approximation result for the spectrum of scattered electrons due to bremsstrahlung, from a static spherically symmetric distribution of charge and magnetic moment.

In general, the functional form of $F_{\rm ch}(x)$ and $F_{\rm mag}(x)$ requires that the integration over x in Eq. (11) be done numerically. However, for certain choices of form factors, the integral may be done analytically. Here we

[10] J. D. Bjorken (unpublished); see also Y. S. Tsai, Ref. 2.
[11] J. M. Jauch, Helv. Phys. Acta **13**, 451 (1940); J. H. Scofield (private communication).
[12] J. D. Walecka and R. H. Pratt, HEPL-272 (unpublished).
[13] G. Racah, Nuovo Cimento **11**, 476 (1934).
[14] P. T. McCormick, D. G. Keiffer, and G. Parzen, Phys. Rev. **103**, 29 (1956).
[15] L. I. Schiff, Phys. Rev. **87**, 750 (1952).
[16] J. I. Friedman, Phys. Rev. **116**, 1257 (1959); W. C. Barber, F. Berthold, G. Fricke, and F. E. Gudden, *ibid.* **120**, 2081 (1960).
[17] W. C. Barber, Ref. 1.
[18] G. H. Rawitscher, Phys. Rev. **101**, 423 (1956).

consider the case of point distributions (for which $|F_{\mathrm{ch}}|^2=|F_{\mathrm{mag}}|^2=1$) to provide a reference spectrum for the finite nuclear-size effects to be discussed in Sec. III. The point results also give an indication of the relative importance of charge and magnetic bremsstrahlung, i.e., bremsstrahlung due to interactions with the charge and magnetic-moment distributions of the nucleus, respectively). For this case the integral of $R_{\mathrm{ch}}x^{-2}$ yields the same result as obtained by previous authors.[13,14] We simply note that $\frac{1}{4}\int_{x_{\min}}^{x_{\max}} R_{\mathrm{ch}}x^{-1}dx$ equals the right-hand side of Eq. (2) of Ref. 14 with $m_e=1$. The corresponding result for the magnetic term is

$$\frac{1}{4}\int_{x_{\min}}^{x_{\max}} R_{\mathrm{mag}}\frac{dx}{x}=\frac{1}{a}\ln\left(\frac{a+k}{a-k}\right)-\frac{1}{k\alpha^2}[\alpha^2+2\alpha(EE_0-1)-2k^2] +\frac{E_0^2+E^2+\alpha-1+2\alpha^{-1}(EE_0-1)}{k[\alpha(\alpha+2)]^{1/2}}\ln\{1+\alpha+[\alpha(\alpha+2)]^{1/2}\} +\frac{\alpha(\alpha-1)+E(E_0+E)-2}{p\alpha^2}\ln(E+p)-\frac{\alpha(\alpha-1)+E_0(E_0+E)-2}{p_0\alpha^2}\ln(E_0+p_0). \quad (14)$$

In the limit of high energies and large scattering angles, E_0, $E\gg1$, $\Theta\gg E_0^{-1}$, which is certainly valid for most electron-scattering experiments, these expressions simplify greatly.[14] One obtains

$$d\sigma=d\sigma_{\mathrm{ch}}+d\sigma_{\mathrm{mag}}, \quad (15)$$

where

$$d\sigma_{\mathrm{ch}}=\frac{\bar{\phi}}{2\pi}\frac{p}{p_0}\frac{dE}{k}\frac{d\Omega_p}{E_0^2}[A_{\mathrm{ch}}(\gamma,\Theta)\ln 2E_0+B_{\mathrm{ch}}(\gamma,\Theta)], \quad (16)$$

$$d\sigma_{\mathrm{mag}}=\frac{\bar{\phi}}{2\pi}\frac{p}{p_0}\frac{dE}{k}d\Omega_p\left(\frac{2\mu}{Ze}\right)^2\left(\frac{I+1}{3I}\right)[A_{\mathrm{mag}}(\gamma,\Theta)\ln 2E_0+B_{\mathrm{mag}}(\gamma,\Theta)], \quad (17)$$

$$A_{\mathrm{ch}}(\gamma,\Theta)=\frac{(1+\gamma^2)^2\cos^2\frac{1}{2}\Theta}{2\gamma^3\sin^4\frac{1}{2}\Theta}, \quad (18)$$

$$A_{\mathrm{mag}}(\gamma,\Theta)=\frac{(1+\gamma^2)(1+\sin^2\frac{1}{2}\Theta)}{\gamma\sin^2\frac{1}{2}\Theta}, \quad (19)$$

$$B_{\mathrm{ch}}(\gamma,\Theta)=\frac{1+\gamma^2}{2\gamma^3}\left\{\frac{\cos^2\frac{1}{2}\Theta}{\sin^4\frac{1}{2}\Theta}\ln\gamma-\frac{1+\gamma^2}{2\sin^4\frac{1}{2}\Theta}+\frac{1-\gamma+\gamma^2}{\sin^2\frac{1}{2}\Theta}+\left(\frac{\gamma}{\sin^4\frac{1}{2}\Theta}-\frac{2\gamma^2}{(1+\gamma^2)\sin^2\frac{1}{2}\Theta}\right)\ln(\sin^2\tfrac{1}{2}\Theta)\right\}, \quad (20)$$

$$B_{\mathrm{mag}}(\gamma,\Theta)=\frac{1-\gamma}{\gamma}\ln\gamma-\frac{1+\sin^2\frac{1}{2}\Theta}{\sin^2\frac{1}{2}\Theta}+\sigma\ln\left(\frac{1+\sigma}{1-\sigma}\right)+\left(1+\frac{1+\gamma^2}{2\gamma\sin^2\frac{1}{2}\Theta}\right)\ln(\gamma\sin^2\tfrac{1}{2}\Theta). \quad (21)$$

In these expressions,

$$\gamma=E/E_0 \quad\text{and}\quad \sigma=(1-\gamma)[(1-\gamma)^2+4\gamma\sin^2\tfrac{1}{2}\Theta]^{-1/2}.$$

The ratio of the magnetic to the charge elastic bremsstrahlung for the point case [Eq. (17) divided by Eq. (16)] is

$$\frac{d\sigma_{\mathrm{mag}}}{d\sigma_{\mathrm{ch}}}=\left(\frac{\lambda}{Z}\right)\left(\frac{I+1}{3I}\right)\left(\frac{E_0}{M_p}\right)^2\frac{A_{\mathrm{mag}}\ln 2E_0+B_{\mathrm{mag}}}{A_{\mathrm{ch}}\ln 2E_0+B_{\mathrm{ch}}}, \quad (22)$$

where we have put $\mu=\lambda e/2M_p$. As expected, the magnetic bremsstrahlung is most appreciable at $\Theta=180°$, where $A_{\mathrm{ch}}=0$, and becomes less important as Z increases. It is interesting to note that for fixed γ and Θ as $E_0\to\infty$ the magnetic bremsstrahlung becomes relatively more and more important. In Fig. 1 we illustrate the relative contributions of the point charge and point magnetic moment for $E_0=54$ MeV and $\Theta=180°$ using Eqs. (16) and (17). The contribution of the magnetic bremsstrahlung is large for scattered electrons undergoing small energy loss, i.e., near the elastic peak from magnetic scattering. The relative importance of the magnetic bremsstrahlung can be inferred from Fig. 1 by multiplying the ratio of the two curves by the factor $(\lambda/Z)^2(I+1)/3I$, which in the case of Al^{27} is ≈0.037 yielding a ratio $d\sigma_{\mathrm{mag}}/d\sigma_{\mathrm{ch}}\approx0.026$ for $\gamma=0.7$ and ≈1.75 at $\gamma=0.95$. (For Cu these percentages are reduced by a factor of 10.) In passing, it may be noted that in the limit $\gamma\to1$, Eq. (22) becomes the same as the corresponding ratio for elastic scattering, as expected.

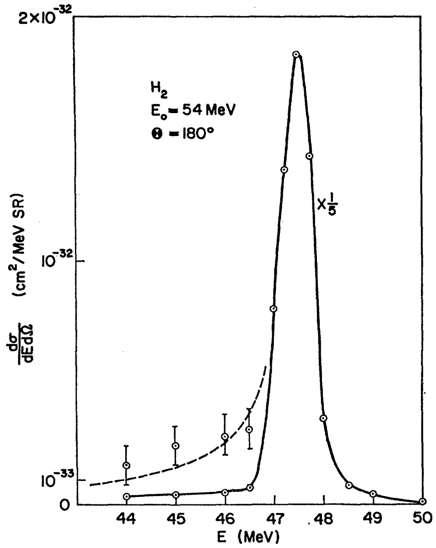

FIG. 2. Elastic peak in electron scattering from hydrogen [J. Goldemberg (unpublished)]. The dashed curve is the approximate analytic result for a point magnetic moment given by Eq. (17).

Recently, magnetic bremsstrahlung in electron-proton scattering has been observed for the first time. Figure 2 shows the results of a measurement of the elastic peak in hydrogen for an incident energy of 54 MeV and a scattering angle of 180°.[19] The experimental points have not been corrected for magnetic bremsstrahlung. The dashed curve represents our approximate theoretical result for the elastic magnetic bremsstrahlung given by Eq. (17). (At these energies the effects of the finite size of the proton are small.) We have interpreted γ as the fraction of the peak energy, i.e., we replace E_0 by the peak energy (the effects of recoil are about 10%). It is encouraging that Eq. (17) agrees fairly well with the data.[20]

Finally, in Fig. 3 we give some indication of how the relative importance of charge and magnetic bremsstrahlung depends on the scattering angle Θ. The magnetic effects begin to be important for angles bigger than 150° for electrons scattered with small energy loss.

III. REPRESENTATIVE NUMERICAL RESULTS

We return now to the discussion of Eq. (11) for the case of general form factors, for which numerical integration is necessary. The functions R_{ch} and R_{mag} are generally sharply peaked about two particular values of x, namely

$$x_a=\alpha+k(k+p-p_0\cos\Theta),$$
$$x_b=\alpha+k(k+p\cos\Theta-p_0).$$

These correspond to values of q for which the photon is emitted parallel to the scattered or incident electrons, respectively. It can be seen from Eqs. (2) and (3) that the differential cross section can become large for these values since at relativistic energies the denominators $(kE-\mathbf{p}\cdot\mathbf{k})$ and $(kE_0-\mathbf{p}_0\cdot\mathbf{k})$ are very small when $\mathbf{k}$ is parallel to $\mathbf{p}$ or $\mathbf{p}_0$. The dominant terms in the totally differential cross section give rise to terms in Eq. (11) which behave like $X^{-1/2}$ or $X_0^{-1/2}$, that is like

$$X^{-1/2}=\{[p(x-x_a)+k(E-p)(p_0\cos\Theta-p)]^2+p_0^2k^2\sin^2\Theta\}^{-1/2}$$

or

$$X_0^{1/2}=\{[p_0(x-x_b)+k(E_0-p_0)(p_0-p\cos\Theta)]^2+p^2k^2\sin^2\Theta\}^{-1/2}.$$

A sizable contribution to the integral in Eq. (11) may arise from a small region in x about x_a or x_b, and this is, in fact, the basis of Schiff's[15] approximation. Thus, one might try to separate the contribution to the integral due to the "peaks" from the remainder, as do Maximon and Isabelle,[8] and integrate over the peaks analytically using the high-energy approximation. However, as Maximon and Isabelle point out, the "background" integral which remains is never negligible, being of the same order of magnitude as the integral over the "peaks;" in some cases it is actually the dominant contribution. Moreover, the "background" integral must still be evaluated numerically,[8] and it is not much easier to compute than the entire integral.

We have therefore proceeded directly to evaluate the integral in Eq. (11) numerically. Concerning the machine program we will only mention two points. First, it is necessary to use a double-precision subroutine to compute the integrands, i.e., R_{ch} and R_{mag}. Second, it is desirable to divide the full range of x into smaller intervals, two of which surround the values x_a and x_b, and to integrate over each smaller interval separately.

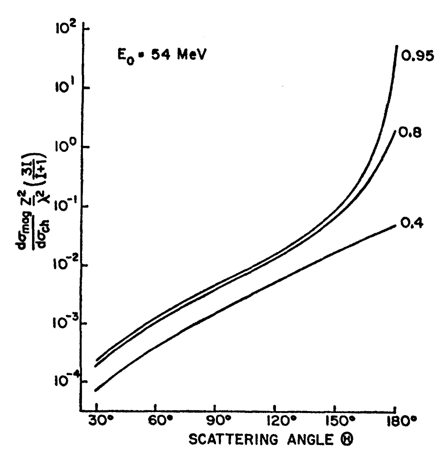

FIG. 3. Dependence on the scattering angle of the ratio of the point magnetic moment to the point-charge bremsstrahlung. The three curves are for $\gamma=E/E_0=0.4$, 0.8, and 0.95.

[19] J. Goldemberg (unpublished).

[20] More exact calculations for electron-proton bremsstrahlung exist, which do not neglect proton recoil: R. A. Berg and C. N. Lindner, Phys. Rev. **112**, 2072 (1958); Nucl. Phys. **26**, 259 (1961); Y. S. Tasi, Ref. 2.

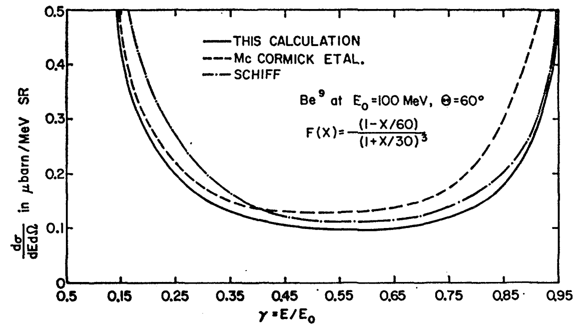

FIG. 4. Comparison of the electron spectrum due to bremsstrahlung as determined by the present calculation with that of previous work. Here, $x=(q\times 2.75\mathrm{F})^2$.

Figure 4 illustrates the differences between the present calculation and previous ones for the case of bremsstrahlung from Be^9 at an incident energy of 100 MeV and a scattering angle of 60°. For this choice of E_0 and Θ the magnetic bremsstrahlung can be neglected. We used a hollow exponential type of form factor

$$F_{\mathrm{ch}}(q^2)=[1-(qR)^2/60][1+(qR)^2/30]^{-3},$$

with $R=2.75$ F, which fits the elastic scattering from Be^9 in this range very well.[21] The difference between the present calculation and that of McCormick *et al.*[14] (point charge) is due to the finite nuclear size; the approximate integration of Schiff[15] (modified to include the form factor[16]) is also shown.

The effect of finite nuclear extension in reducing the radiative tail is clearly shown in Fig. 5, where we have plotted the ratio of cross section, given by Eq. (11), to the point cross section, for some spinless nuclei (the situation is not complicated by the presence of magnetic bremsstrahlung). The values of E_0 (70 MeV) and Θ (180°) correspond to recent experiments at Stanford.[22] For simplicity, we have chosen a form factor corresponding to a trapezoidal charge distribution for all three nuclei, with values of the half-radius and skin thickness taken from Table 3 of Herman and Hofstadter.[23]

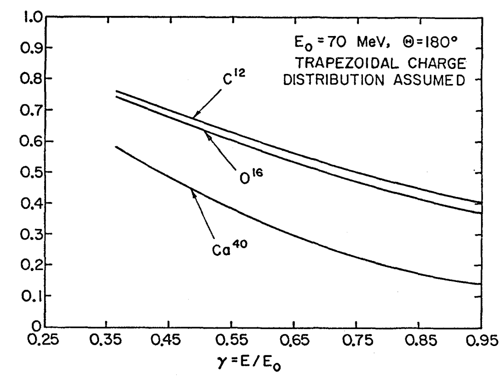

FIG. 5. Ratio of the cross section calculated with form factors (determined from elastic scattering) to the point-charge result, for some spinless nuclei.

Finally, in Fig. 6, we illustrate the effect of the contribution of magnetic bremsstrahlung. Inelastic scattering experiments have been performed on Li^7 at 180° using 41.5-MeV incident electrons.[24] This nucleus has a large magnetic moment[25] and a low Z [the factor $(\lambda/Z)^2(I+1)/3I$ is approximately 0.65] so we expect magnetic effects to be important. As a first approximation we may set $|F_{\mathrm{ch}}|^2=|F_{\mathrm{mag}}|^2$ and assume a trapezoidal form factor with parameters taken from Herman and Hofstadter.[23] Besides the differential cross section, the separate contributions of $d\sigma_{\mathrm{ch}}$ and $d\sigma_{\mathrm{mag}}$ are also shown in Fig. 6. It is seen that the ratio $d\sigma_{\mathrm{mag}}/d\sigma_{\mathrm{ch}}$ is about 0.24 at $\gamma=0.7$ and rises up to 8.0 at $\gamma=0.95$. [This compares with ratios for the point case from Eq. (22) of 0.25 and 17.4, respectively.]

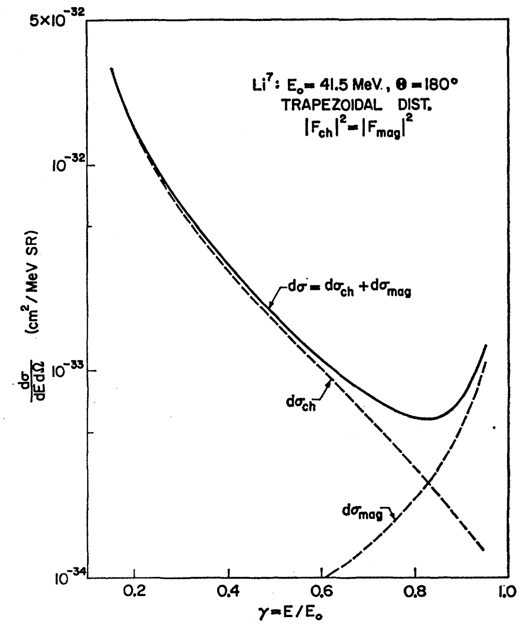

FIG. 6. Spectrum of scattered electrons due to bremsstrahlung from Li^7, showing contribution of the charge and magnetic-moment distributions.

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